DISCRETIZATION PROCESS IN FINDING NUMERICAL SOLUTIONS OF HEAT TRANSFER PROBLEMS IN LIVING SYSTEMS AND OTHERS

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Abstract: Heat transfer problems are very important in the field of engineering as well as in biology. For heat transfer problems generally analytical solutions are given. Analytical solutions which allow us to determine the exact temperature distribution are only available for limited ideal cases. When analytical solutions are not available, graphical solutions are used for complex heat transfer problems. Advances in numerical computing now allow us for complex heat transfer problems to be solved rapidly on computers. At present following techniques are commonly used (i) Finite-Difference, (ii) Finite element method and (iii) Finite volume method [1],[2].In general these methods are routinely used to solve problems in heat transfer, fluid dynamics, stress analysis, electrostatics and magnetic etc. Numerical techniques result in an approximate solution, however the error can be made very small. Properties (e.g., temperature) are determined at discretepoints in the region of interest-these are referred to as nodal pointsor nodes.

In this paper we have discussed the basics of discretization process and the need of numerical solutions.

Keywords: Numerical solution, Mesh, Discretization, Finite difference, Finite element, Finite volume.

Introduction: Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very use full, especially when the geometry of the object of interest is irregular, of the boundary conditions are nonlinear. In heat transfer analysis, some bodies are considered as a 'lump'. In a 'lump' interior temperature remains constant during heat transfer. The temperature of such bodies can be taken as a function of time only [3]. Freezing of food, cooking of food, boiling of eggs are some examples of heat transfer problems in daily life. The growth rate of microorganism in a food product in environmental temperature is another example of heat transfer. In numerical analysis, three different approaches are commonly used; the finite difference, the finite volume and the finite element methods. Brief descriptions of the three methods are as follows:

Why Numerical Methods? The ready availability of high-speed computers and easy-to-use powerful software packages has had a major impact on engineering education and practice in recent years. Engineers in the past had to rely on analytical skills to solve significant engineering problems, and thus they had to undergo a rigorous training in mathematics. Today's engineers, on the other hand, have access to a tremendous amount of computation power under their fingertips, and they results, But they also need to understand how calculations are performed by the computers to develop an awareness of the process involved and the limitations, white avoiding any possible pitfalls[4].

Limitations Analytical solution methods are limited to highly simplified problems in simple geometries. The geometry must be such that its entire surface can be described mathematically in a coordinate system by setting the variables equal to constants. That is, it must fit into a coordinates system perfectly with nothing sticking out or in. In the case of one-dimensional heat conduction in a solid sphere radiusr_0, for example, the entire outer surface can be described by $r = r_0$. Likewise, the surface of a finite solid cylinder of radius $r_0$and height H can be described by $r = r_0$ for the side surface and z = 0 and z = H for the bottom and top surfaces, respectively. Even minor complication in geometry can make an analytical solution impossible. For example, a spherical object with an extrusion likes a handle at some location is impossible to handle analytically since the boundary conditions in this case cannot be expressed in any familiar coordinate system. In dization process of food in stomach is very difficult for modeling,because it is an example of lump which flows inside the stomach.

Better Modeling: We mentioned earlier that analytical solutions are exact solutions since they do not involve any approximations. But this statement needs some clarification. Distinction should be made between an actual real-world problem and the mathematical model that is an idealized representation of it. The solutions we get are the solutions of mathematical models, and the degree of applicability of these solutions to the actual physical problems depends on the accuracy of the model. An "approximate" solution of a realistic model of a physical problem is usually more accurate than the "exact" solution of a crude mathematical model.

When attempting to get an analytical solution to a physical problem. There is always the tendency to oversimplify the problem to make the mathematical model sufficiently simple to warrant an analytical solution. Therefore, it is common practice to ignore any effects that cause mathematical complications such as a nonlinearities in the differential equation or the boundary conditions. So it comes as no surprise that nonlinearities such as temperature dependence of thermal conductivity and the radiation boundary conditions are seldom considered in analytical solutions. A mathematical model intended for a numerical solution is likely to represent the actual problem better. Therefore, the numerical solution of heat transfer problems has now become the norm rather than the exception even when analytical solutions are available.

Flexibility: Heat transfer problems often require extensive parametric studies to understand the influence of some variables on the solution in order to choose the right set of variables and to answer some “what-if” questions. This is an iterative process that is extremely tedious and time-consuming if done by hand. Computers and numerical methods are ideally suited for such calculations, and a wide range of related problems can be solved by minor modifications in the code or input variables. Today it is almost unthinkable to perform any significant optimization studies in engineering without the power and flexibility of computers and numerical methods.

Complications: Some problems can be solved analytically, but the solution procedure is so complex and the resulting solution expressions so complicated that it is not worth all that effort. With the exception of steady one-dimensional or transient lumped system problems, all heat conduction problems result in partial differential equation beyond that acquired at the undergraduate level, such as orthogonality, Eigen values, Fourier and Laplace transforms, Bessel and Legendre functions, and infinite series. In such cases, the evaluation of the solution, which often involves double or triple summations of infinite series at a specified point, is a challenge in itself.

Human Nature: As human beings, we like to sit back and make wishes, and we like our wishes to come true without much efforts. The invention of TV remote controls made us feel like kings in our homes since the commands we give in our comfortable chairs by pressing buttons are immediately carried out by the obedient TV sets. After all, what good is cable TV without a remote control? We certainly would love to continue being the king in our little cubicle in the engineering office by solving problems at the press of a button on a computer (until they invent a remote control for the computers, of course). Well this might have been a fantasy yesterday, but it is a reality today. Practically all engineering offices today are equipped with high-powered computers with sophisticated software packages, with impressive presentation-style colorful output in graphical and tabular form. Besides, the results are as accurate as the analytical results for all practical purposes. The computers have certainly changed the way engineering is practiced.

Overview: Discretization is a cornerstone of numerical techniques i.e. numerical solution. An analytical solution to a partial differential equation gives us the value of f as a function of the independent variables (x_y, z, t). On the other hand, the numerical solution provides us the value of f at a discrete number of points in the domain. These points are called grid points, or sometimes as nodes or cell centroids, depending on the method. The process of converting our governing transport equation into a set of equations for the discrete values of f is called the discretization process and the specific methods employed to bring about this conversion are called discretization methods[5].

The discrete values of f are typically described by algebraic equations relating the values at grid points to each other. The development of numerical methods focuses on both the derivation of the discrete set of algebraic equations, as well as a method for their solution. In arriving at these discrete equations for f we will be required to assume how f varies between grid points i.e., to make profile assumptions. Most widely used methods for discretization require local profile assumptions. That is, we prescribe how f varies in the local neighborhood surrounding a grid point, but not over the entire domain.

The conversion of a differential equation into a set of discrete algebraic equations requires the discretization of space. This is accomplished by means of mesh generation. Mesh generation divides the domain of our interest into elements or cells, and associates with each element or cell one or more discrete values of f .

Since our aim is to get an answer to the original differential equation, it is appropriateto check whether our algebraic equation set really gives us this. When the number of grid points is small, the departure of the discrete solution from the exact solution is expected to be large. A well-behaved numerical scheme will tend to the exact solution as the number of grid points is increased. The rate at which it tends to the exact solution depends on the type of profile assumptions made in obtaining the discretization. No matter what discretization method is employed, all well-behaved discretization methods should tend to the exact solution when a large enough number of grid points are employed.

a).Mesh Terminology: The physical domain is discretized by meshing or gridding .The fundamental unit of the mesh is the cell (sometimes called the element). Associated with each cell is the cell centroid. A cell is surrounded by faces, which meet at nodes or vertices. In three dimensions, the face is a surface surrounded by edges. In two dimensions, faces and edges are the same.

b) Types: There are several types of meshes in practice. Some are described below.

Regular and Body-fitted Meshes

Sometimes our interest lies in analyzing domains which are regular in shape i.e. rectangles, cubes, cylinders, spheres etc. These shapes can be meshed by regular grids. The grid lines are orthogonal to each other, and conform to the boundaries of the domain. These meshes are also sometimes called orthogonal meshes.

Sometimes the domains of our interest are irregular in shape. In such conditions gridlines are not necessarily orthogonal to each other, and curve to conform to the irregulargeometry. If regular grids are used in these geometries, stair stepping occurs at domainboundaries. When the physics at the boundary are important in determining the solution,

e.g., in flows dominated by wall shear, such an approximation of the boundary may not be acceptable.

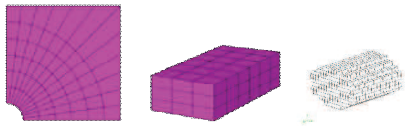

Fig.1. Structured, Block Structured, and Unstructured Meshes

The meshes shown in Fig.1 are examples of structured meshes. Here, every interior vertex in the domain is connected to the same number of neighbor vertices. Fig.2. shows a block-structured mesh. Here, the mesh is divided into blocks, and the mesh within each block is structured.

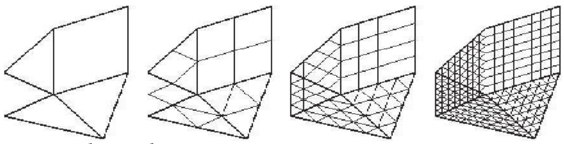

Fig.2. Block-structured mesh

However, the arrangement of the blocks themselves is not necessarily structured. Fig.3 shows an unstructured mesh. Here, each vertex is connected to an arbitrary number of neighbor vertices. Unstructured meshes impose fewer topological restrictions on the user, and as a result, make it easier to mesh very complex geometries.

Conformal and Non-Conformal Meshes: There are two conditions for a conformal mesh (6). First the intersection between any two elements is a sub element of both: a face, an edge, a node or nothing (the void set). Second the maximal dimensional shared element must be only one and complete. When the vertices of a cell or element may fall on the faces of neighboring cells or elements.Example of a non-conformal mesh is shown in Fig.3. Here, the vertices of acell or element may fall on the faces of neighboring cells or elements. In contrast, themeshes in Figures are conformal meshes.

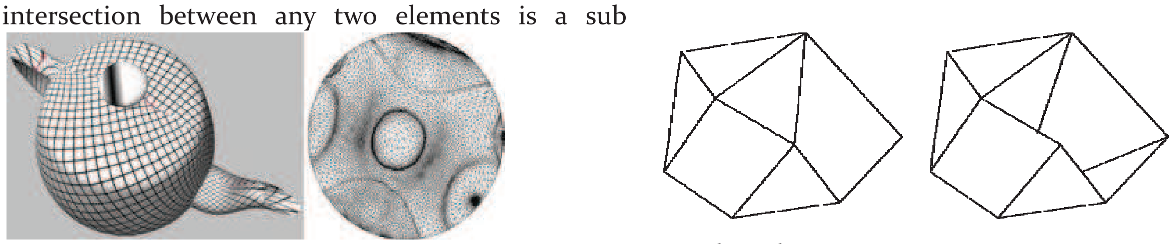

Fig.3. Unstructured mesh

Cell Shapes: Meshes may be constructed using a variety of cell shapes. The most widely used are quadrilaterals and hexahedra. Methods for generating good-quality structured meshes for quadrilaterals and hexahedra have existed for some time now. Though mesh structure imposes restrictions, structured quadrilaterals and hexahedra are well-suited for flows with a dominant direction, such as boundary-layer flows. More recently, as computational fluid dynamics is becoming more widely used for analyzing industrial flows, unstructured meshes are becoming necessary to handle complex geometries. Here, triangles and tetrahedra are increasingly being used, and mesh generation techniques for their generation are rapidly reaching maturity.

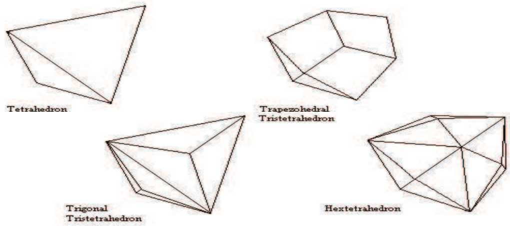

Fig.4.Types of cell shapes

Cell Shapes: (a) Triangle, (b) Tetrahedron, (c) Quadrilateral, (d) Hexahedron, (e) Prism, and (f) Pyramid

purpose techniques for generating unstructured hexahedra. Another recent trend is the use of hybrid meshes. For example, prisms are used in boundary layers, transitioning to tetrahedra in the free-stream. In this book, we will develop numerical methods capable of using all these cell shapes.

Node-Based and Cell-Based Schemes: Numerical methods which store their primary unknowns at the node or vertex locations are called node-based or vertex-based schemes. Those which store them at the cell centroid, or associate them with the cell, are called cell-based schemes. Finite element methods are typically node-based schemes, and many finite volume methods are cell-based. For structured and block-structured meshes composed of quadrilaterals or hexahedra, the number of cells is approximately equal to the number of nodes, and the spatial resolution of both storage schemes is similar for the same mesh. For other cell shapes, there may be quite a big difference in the number of nodes and cells in the mesh. For triangles, for example, there are twice as many cells as nodes, on average. This fact must be taken into account in deciding whether a given mesh provides adequate resolution for a given problem. From the point of view of developing numerical methods, both schemes have advantages and disadvantages, and the choice will Discretization Methods.

The Finite Difference Method (FDM): This is the oldest method for numerical solution of PSEs, introduced by Euler in the 18th century. It's also the easiest method to use for simple geometries. The starting point is the conservation equation in differential form. The solution domain is covered by grid. At each grid point, the differential equation id approximated by replacing the partial derivatives by approximations in terms of the nodal values of the functions. The result is one algebraic equation per grid node, in which the variable value at that and a certain number of neighbor nodes appear as unknowns.

In principle, the FD method can be applies to any grid type. However, in all applications of the FD method known, it has been applied to structured grids. Taylor series expansion or polynomial fitting is used to obtain approximations to the first and second derivatives of the variables with respect to the coordinates. When necessary, these methods are also used to obtain variable values at locations other than grid nodes (interpolation).

On structured grids, the FD method is very simple and effective. It is especially easy to obtain higher-order schemes on regular grids. The disadvantage of FD methods is that the conservation is not enforced unless special care is taken. Also, the restriction to simple geometries is a significant disadvantage

Finite Volume Method (FVM): The FV method uses the integral form of the conservation equations as its starting point. The solution domain is subdivided into a finite number of contiguous control volumes (CVs), and the conservation equations are applied to each CV. At the centroid of each CV lies a computational node at which the variable values are to be calculated. Interpolation is used to express variable values at the CV surface in terms of the nodal (CV-center) values. As a result, one obtains an algebraic equation for each CV, in which a number of neighbor nodal values appear. The FVM method can accommodate any type of grid when compared to FDM, which is applied to only structured grids. The FVM approach is perhaps the simplest to understand and to program. All terms that need be approximated have physical meaning, which is why it is popular.

Finite Element Method (FEM): The FE method is similar to the FV method in many ways. The domain is broken into a set of discrete volumes or finite elements that are generally unstructured; in 2D, they are usually triangles or quadrilaterals, while in 3D tetrahedra or hexahedra are most often used.

The distinguishing feature of FE methods is that the equations are multiplied by a weightfunction before they are integrated over the entire domain. In the simplest FE methods, thesolution is approximated by a linear shape function within each element in a way that guarantees continuity of the solution across element boundaries. Such a function can beconstructed from its values at the corners of the elements. The weight function is usually ofthe same form. This approximation is then substituted into the weighted integral of the conservation law and the equations to be solved are derived by requiring the derivative of the integral with respect to each nodal value to be zero; this corresponds to selecting the best solution within the set of allowed functions (the one with minimum residual). The result is a set of non-linear algebraic equations. An important advantage of finite element methods is the ability to deal with arbitrary geometries. Finite element methods are relatively easy to analyze mathematically and can be shown to have optimality properties for certain types of equations. The principal drawback, which is shared by any method that uses unstructured grids, is that the matrices of the linearized equations are not as well structured as those for regular grids making it more difficult to find efficient solution methods.

Summary: We note the following about the discretization process.

1. The process starts with the statement of conservation over the cell. We then find cell values of □□which satisfy this conservation statement. Thus conservation is guaranteed for each cell, regardless of mesh size.
2. Conservation does not guarantee accuracy, however. The solution for*f*□□may beinaccurate, but conservative.
3. The cell balance iswritten in terms of face fluxes. The gradient of f□□ must therefore be evaluated at the faces of the cell.
4. The profile assumptions for f□□and S need not be the same.The disadvantage of FV methods compared to FD schemes is that methods of order higher than second are more difficult to develop in 3D. This is due to the fact that the FV approach requires two levels of approximation: interpolation and integration.

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